Datum Definition and its Influence on the Reliability of Geodetic Networks

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Summary

This paper attempts to study the effects of datum definition on the reliability of geodetic networks. Particular attention is paid to the geometry of datum points, defined here as the number and distribution of the datum points, and to its effect on the external reliability of the network points.

While the internal reliability of a network is independent of datum definition, the external reliability depends heavily on the distribution of datum points. This paper presents a new perspective and describes relevant parameters that enable to define and quantify the influence of the datum on the external reliability of geodetic networks.

Following an introduction of the concept of geodetic networks' reliability and a development of theoretical tools, the paper presents results of numerical experiments carried out with a schematic horizontal GPS network. These results indicate that the reliability of the adjusted coordinates in a geodetic network depends on the geometrical distribution of the points that define the datum of the network.

Zusammenfassung

In diesem Beitrag wird die Auswirkung der Datumsdefinition auf die Zuverlässigkeit geodätischer Netze diskutiert. Speziell wird die Geometrie der Datumspunkte, die über ihre Anzahl und Verteilung definiert ist, und deren Einfluss auf die äußere Zuverlässigkeit der Netzpunkte untersucht.

Während die innere Zuverlässigkeit eines Netzes unabhängig von der Datumsdefinition ist, hängt die äußere Zuverlässigkeit in starkem Maße von der Verteilung der Datumspunkte ab. Dieser Beitrag stellt eine neue Sichtweise des Sachverhalts vor und beschreibt die relevanten Parameter, die die Definition und Quantifizierung des Einflusses des Datums auf die äußere Zuverlässigkeit des geodätischen Netzes ermöglichen.

Nach der Einführung des Konzepts zur Zuverlässigkeit geodätischer Netze und der Herleitung der theoretischen Formalismen werden die Ergebnisse numerischer Experimente, die mit Hilfe eines schematischen GPS-Lagenetzes ausgeführt wurden, vorgestellt. Diese Ergebnisse zeigen, dass die ausgeglichenen Koordinaten in einem geodätischen Netz von der geometrischen Verteilung der Punkte, die das Datum des Netzes definieren, abhängen.

1 Introduction

A geodetic datum is a set of parameters and control points used to mathematically define the size and shape of the earth and the origin and orientation of the coordinate systems used to map the earth. Modern geodetic datums range from flat-earth models used for plane surveying to complex models used for international applications, which describe the size, shape, orientation, gravity field, and angular velocity of the earth. The diversity of datums in use today and the technological advancements that enable global positioning measurements that are accurate to a decimeter level require careful datum selection and conversion between coordinates in different datums.

When discussing geodetic networks, a datum is defined as a subset of network points. In hierarchical networks the datum is defined by the higher order control points. In general, datum points should remain stable and consistent over time. The selection of the datum points is an initial stage in establishing and designing a geodetic control network. Grafarend (1974) defined the datum design problem as the first stage of four in the process of designing a geodetic network, and called it the Zero Order Design.

Geodetic networks should be designed and tested according to three criteria that determine their quality and utility, accuracy, reliability and cost. An ideal network will be one with good accuracy, high reliability and low cost. These criteria can be tested on the part of the network which is being used to define the datum.

The accuracy of the datum and its dependence on the network geometry was investigated by Papo (1999). The experiments reported in his paper proved the effect of datum selection on the accuracy of the points in the network, where a wide geometrical distribution of the datum points was better than a narrow geometrical distribution.

This paper investigates the influence of datum definition on the network reliability, and explores the relationship between the datum points' geometry and the reliability of the network points. Particular attention is paid to GPS networks, since GPS technology is a very effective and predominant tool in constructing geodetic networks.

2 The reliability concept

Reliability is defined as the ability of the network to sense and identify gross errors in the measurements. Baarda (1968) distinguishes between »internal reliability« and »external reliability«. The internal reliability of a control network measures the marginally undetectable errors in the measurements, while the external reliability measures the effect of undetected gross errors on the network coordinates and on quantities computed from them.

2.1 Internal reliability

The w test introduced by Baarda (1968) is used in assessing the internal reliability. Let us define $\hat{d}_i$ as the difference between an observed (ℓ_i) and a computed ($\hat{\ell}_i^c$) value, $\hat{d}_i = \ell_i - \hat{\ell}_i^c$ with a standard deviation $\sigma_{\hat{d}_i}$. The i-th estimated quantity $\hat{\ell}_i^c$ computed from the parameters following from the adjustment computation of n observations ℓ except ℓ_i. The test statistic w_i is defined as:

$$w_i = \frac{\hat{d}_i}{\sigma_{\hat{d}_i}} \; . \tag{1}$$

Baarda showed that w_i is normally distributed with a zero mean and unit variance when the observations are without gross errors. Let Δ_{ℓ_i} denote a single gross error connected to observation i. In this case the mean of the normal distribution is:

$$\delta_i = \frac{\Delta_{\ell_i}}{\sigma_{\hat{d}_i}} \; . \tag{2}$$

The upper boundary of δ_i is δ_i^u with probability levels α and β (α being the level of significance, β being the test power). Therefore, the maximum size of a gross error that could contaminate the i-th observation is:

$$\Delta_i^u = \delta_i^u \sigma_{\hat{d}_i} \; . \tag{3}$$

The vector of adjusted measurements $\hat{L}$ is a linear function of the design matrix A and the adjusted parameters $\hat{x}$:

$$\hat{L} = A\hat{x} = A(A^T P A)^+ A^T PL = HL \; . \tag{4}$$

When free geodetic networks are considered the Moore-Penrose inverse ($^+$) is commonly used. The square matrix H is termed the »hat« matrix. Equation (4) can be normalized by pre-multiplying the matrix A and the vector L by S. S is defined as the square root matrix of the weight matrix P. If C is the matrix of eigenvectors and D the diagonal matrix of eigenvalues of P, then $S = \sqrt{P} = C\sqrt{D}C^T$. S exists for any P that is positive semi-definite and symmetric, which is always the case with the weight matrix. Now equation (4) can be written in its normalized form, were script letters denotes normalized matrices:

$$\hat{\mathcal{L}} = \mathcal{A}(\mathcal{A}^T \mathcal{A})^+ \mathcal{A}^T \mathcal{L} = \mathcal{H}\mathcal{L} \; . \tag{5}$$

The idempotent, symmetric matrix $\mathcal{H}$ is also termed a projector. Let $\hat{h}_i$ be the i-th diagonal element of $\mathcal{H}$. For a projector, the sum of squares of the entries of each row is equal to the row's diagonal entry: $\sum_k \hat{h}_{ik}^2 = \hat{h}_i$. The diagonal elements of matrix $\mathcal{H}$ fulfill the inequality $0 \le \hat{h}_1 \le 1$.

Even-Tzur (2000) has shown that $\sigma_{\hat{d}_i}$ can be presented as a function of $\hat{h}_i$:

$$\sigma_{\hat{d}_i} = \sigma_0 \Big/ \sqrt{1 - \hat{h}_i} \tag{6}$$

while σ_0^2 is the a-priori variance of unit weight. Now we use $\sigma_{\hat{d}_i}$ in (3) to calculate the maximum size of a gross error Δ_i^u that could contaminate the i-th observation. For a particular adjustment σ_0 is fixed variable and δ_i^u is a constant that depends on α and β, therefore the matrix $\mathcal{H}$ serves as a primary tool in the internal reliability analysis. Huber (1981) recommended avoiding elements of $\hat{h}_i$ being larger than 0.5, to preclude excessive influence of a single measurement on the adjusted parameters and to ensure well-balanced and controlled solutions. Because of the form of the design matrix in GPS measurements, the matrix $\mathcal{H}$ is independent of the network points' configuration (Even-Tzur and Papo 1996). The internal reliability of a GPS network is independent of the geometric distribution of the network points. It depends on the configuration of the GPS vectors and their precision.

We can define $r_i = 1 - \hat{h}_i$ as the degree of freedom for measurement i. Similarly, the degree of freedom for a single point in the network can be defined as the sum of all the degrees of freedom for the measurements attached to the point. Assuming that k measurements are related to a point p in the network, then the degree of freedom for a single point r_p is equal to:

$$r_p = k - \sum_{i=1}^{k} \hat{h}_i \; . \tag{7}$$

The degree of freedom for a single point could serve as a key number in the assessment of network reliability.

2.2 External reliability

External reliability measures the influence of undetected gross errors on the estimation of coordinates. Let a gross error in observation i be denoted as Δ_i^u and $\Delta_{\ell_i} = \begin{pmatrix} 0 & 0 & . & . & \Delta_i^u & . & . & 0 \end{pmatrix}^T$. For the gross error vector Δ_{ℓ_i} we compute the changes $\Delta\hat{X}_i$ of the adjusted coordinates in a given reference system:

$$\Delta\hat{X}_i = (\mathcal{A}^T \mathcal{A})^+ \mathcal{A}^T \Delta_{\ell_i} = N^+ \mathcal{A}^T \Delta_{\ell_i} . \tag{8}$$

$\Delta\hat{X}_i$ is a $u \times 1$ vector which depends on the reference system.

According to Baarda (1968), the global external reliability can be measured by:

$$\lambda_{\Delta\hat{x}_i}^2 = \frac{1}{\sigma_0^2} (\Delta\hat{X}_i)^T N(\Delta\hat{X}_i) \; . \tag{9}$$

When substituting equation (8) into equation (9):

$$\lambda^2_{\Delta\hat{x}_i} = \left(\delta^u_i\right)^2 \frac{\hbar_i}{1 - \hbar_i} \ . \tag{9'}$$

The global external reliability denotes the impact of a single gross error on the adjusted coordinates.

When attempting to derive an expression for describing the impact of each marginally detectable gross error on all estimable coordinates, we can define M as the diagonal matrix for the n observations while

$$\text{diag}(M) = \left(\Delta^u_1 \quad \Delta^u_2 \quad . \quad . \quad . \quad \Delta^u_n\right).$$

M can be termed the internal reliability matrix. In case $\mathcal{A}$ is rank deficient due to the need for datum definition, hence

$$\Delta\hat{X}_M = N^+ \mathcal{A}^T M \ . \tag{10}$$

If u is the number of adjusted coordinates in the network and n is the number of observations, then $\Delta\hat{X}_M$ is a $u \times n$ matrix. $\Delta\hat{x}_{ij}$ are the entries of the $\Delta\hat{X}_M$ matrix and each entry represents the effect of gross error of a magnitude Δ^u_i in measurement i on the adjusted coordinate j. Therefore, each column of matrix $\Delta\hat{X}_M$ shows the effect of gross errors of a magnitude Δ^u_i on the adjusted coordinates. The total impact of n marginally detectable gross errors on the adjusted coordinates can be calculated as L^2-norm for each row of matrix $\Delta\hat{X}_M$:

$$\Delta\hat{X} = \begin{bmatrix} \sqrt{\sum_j (\Delta\hat{x}_{1,j})^2} \\ \sqrt{\sum_j (\Delta\hat{x}_{2,j})^2} \\ . \\ \sqrt{\sum_j (\Delta\hat{x}_{i,j})^2} \\ . \\ . \\ \sqrt{\sum_j (\Delta\hat{x}_{3u,j})^2} \end{bmatrix} \quad \text{for} \quad \begin{array}{l} j = 1, 2, ..., n \\ i = 1, 2, ..., u \end{array} \tag{11}$$

The vector $\Delta\hat{X}$ defines the external reliability of the adjusted values. Thus, the global external reliability of the network can be defined as:

$$\lambda^2 = \text{tr}[(\Delta\hat{X}_M)(\Delta\hat{X}_M)^T] \ . \tag{12}$$

λ^2 defines the external reliability of the whole network and is dependent on datum definition. To present the reliability of a group of points we can use equation (12) and find the trace related to these particular points. It can then be used to find the reliability of the datum points, as well as the control points, in the network.

3 Datum definition and its influence on reliability

According to Wolf (1977), we can transform one solution, $\hat{x}$, pertaining to a certain datum into another, $\underline{\hat{x}}$, pertaining to another datum using a similarity transformation. Let I be the identity matrix and G a similarity transformation matrix, also known as Helmert's transformation matrix. Such a transformation is described by:

$$\underline{\hat{x}} = [I + GB]\hat{x} = J\hat{x} \tag{13}$$

while $B = -(G^T G)^{-1} G^T$. $\underline{\hat{x}}$ is the unique solution that yields $\underline{\hat{x}}^T \underline{\hat{x}} \to \min$. Note that J is idempotent: $J = J^2$ and $J = J^T$, hence J is an orthogonal projector.

For GPS networks the size of G is $3u \times 3$ due to the defect of rank 3 in the normal matrix. Since the definition of origin is missing, G looks as:

$$G^T = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 1 & 0 & 0 \\ 0 & 1 & 0 & . & . & . & 0 & 1 & 0 \\ 0 & 0 & 1 & . & . & . & 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

The cofactor matrix for the solution $\hat{x}$ is called Q. Wolf (1977) states that $QG = 0$ (when the datum is defined by all network points), as G spans the null space of $\mathcal{A}$ and $Q = (\mathcal{A}^T \mathcal{A})^+$. Another important property is $\mathcal{A}G = 0$ (Papo 1987).

In accordance with the law of error propagation, the cofactor matrix $\underline{Q}$ of the transformed solution $\underline{\hat{x}}$ is

$$\underline{Q} = JQJ^T \ . \tag{15}$$

The solution $\underline{\hat{x}}$ and its cofactor matrix $\underline{Q}$ are based on a datum defined by all the points in the network. It is the optimal datum since the trace of the cofactor matrix $\underline{Q}$ is minimal (Meissl 1969). However, the application of a datum definition that relies on all the points in the network is not practical. It is hard to find even a single case in which we would like to base the datum of a network on the coordinates of all of its points.

Let P_x be a diagonal matrix with 1 for points that enter the datum definition and 0 for all others. When searching for a solution with $\underline{\hat{x}}^T P_x \underline{\hat{x}} \to \min$ we get:

$$J = I - G(G^T P_x G)^{-1} G^T P_x \ . \tag{16}$$

The projector $G(G^T P_x G)^{-1} G^T P_x$ is no longer a symmetric matrix. For GPS networks the matrix J is independent of the geometrical distribution of the datum points.

3.1 Internal reliability and datum definition

The matrix $\mathcal{H}$ can be defined as $\mathcal{H} = \mathcal{A}Q\mathcal{A}^T$. When applying the similarity transformation on the cofactor matrix Q for investigating if $\mathcal{H}$ depends on the datum definition, we get:

$$
\begin{aligned}
\underline{\mathcal{H}} &= \mathcal{A}\underline{Q}\mathcal{A}^T = \mathcal{A}JQJ^T\mathcal{A}^T \\
&= \mathcal{A}[I - G(G^TP_xG)^{-1}G^TP_x]Q[I - G(G^TP_xG)^{-1}G^TP_x]^T\mathcal{A}^T \\
&= \mathcal{A}[Q - QP_xG(G^TP_xG)^{-1}G^T - G(G^TP_xG)^{-1}G^TP_xQ \\
&\quad + G(G^TP_xG)^{-1}G^TP_xQP_xG(G^TP_xG)^{-1}G^T]\mathcal{A}^T = \mathcal{A}Q\mathcal{A}^T.
\end{aligned}
$$

$$(17)$$

Hence, the matrix $\mathcal{H}$ is independent of the specific datum definition. Since the matrix $\mathcal{H}$ defines the internal reliability, we can conclude that the internal reliability is independent of the datum definition. Since the internal reliability measures the marginal undetectable errors in the measurements, we could intuitively state that the internal reliability is not affected by the datum definition, as datum definition is not based on measurements.

Since $\hat{h}_i$ does not depend on the datum definition, Baarda's measure for the global external reliability (see 9') is independent of the datum choice.

3.2 External reliability and datum definition

As previously defined, Q is the cofactor matrix of $\hat{x}$ and $\underline{Q}$ is the cofactor matrix of $\underline{\hat{x}}$. According to equation (12) the network's global external reliability of $\hat{x}$ is defined as:

$$
\begin{aligned}
\lambda^2 &= tr[(\Delta X_M)(\Delta X_M)^T] = tr[(\Delta X_M)^T(\Delta X_M)] \\
&= tr[M\mathcal{A}QQ\mathcal{A}^TM].
\end{aligned}
$$

$$(12')$$

According to equation (15) the cofactor matrix $\underline{Q}$ of the transform solution $\underline{\hat{x}}$ is:

$$
\underline{Q} = (I + GB)Q(I + B^TG^T).
$$

$$(15')$$

Therefore, the global external reliability of $\underline{\hat{x}}$ is:

$$
\begin{aligned}
\underline{\lambda}^2 &= tr[M\mathcal{A}(I + GB)Q(I + B^TG^T)(I + GB)Q(I + B^TG^T)\mathcal{A}^TM] \\
&= tr[M\mathcal{A}(Q + QB^TG^T + GBQ + GBQB^TG^T) \\
&\quad (Q + QB^TG^T + GBQ + GBQB^TG^T)\mathcal{A}^TM].
\end{aligned}
$$

$$(18)$$

Since $\mathcal{A}G = 0$ we get:

$$
\begin{aligned}
\underline{\lambda}^2 &= tr[(M\mathcal{A}Q + M\mathcal{A}QB^TG^T)(Q\mathcal{A}^TM + GBQ\mathcal{A}^TM)] \\
&= tr[M\mathcal{A}QQ\mathcal{A}^TM + M\mathcal{A}QGBQ\mathcal{A}^TM \\
&\quad + M\mathcal{A}QB^TG^TQ\mathcal{A}^TM + M\mathcal{A}QB^TG^TGBQ\mathcal{A}^TM] \\
&= tr[\mathcal{A}^TMM\mathcal{A}(QQ + QGBQ + QB^TG^TQ + QB^TG^TGBQ)].
\end{aligned}
$$

$$(18')$$

Let us find for which B the network's global external reliability ($\underline{\lambda}^2$) is extremum, by searching for a B that fulfills the equation $\partial[\underline{\lambda}^2]/\partial B = 0$:

$$
\begin{aligned}
\frac{\partial[\underline{\lambda}^2]}{\partial B} &= G^TQ\mathcal{A}^TMM\mathcal{A}Q + G^TQ\mathcal{A}^TMM\mathcal{A}Q \\
&\quad + G^TGBQ\mathcal{A}^TMM\mathcal{A}Q + G^TGBQ\mathcal{A}^TMM\mathcal{A}Q \\
&= 2G^TQ\mathcal{A}^TMM\mathcal{A}Q + 2G^TGBQ\mathcal{A}^TMM\mathcal{A}Q \\
&= 2(G^T + G^TGB)Q\mathcal{A}^TMM\mathcal{A}Q = 0.
\end{aligned}
$$

$$(19)$$

When $B = -(G^TG)^{-1}G^T$ we obtain $\partial[\underline{\lambda}^2]/\partial B = 0$.

Thus the datum with the minimal trace of the cofactor matrix $\underline{Q}$ yields the minimal $\underline{\lambda}^2$ value. This means that the best external reliability is achieved when datum is based on all the network points.

In reality there are no geodetic networks that are based on all of their points, and usually only a small set of points is used. Each datum relies on a different set of points leading to a different external reliability vector ($\Delta\hat{X}$) and a different network global external reliability (λ^2). A reliable datum will be one with a minimal influence of gross errors in the measurements on the adjusted coordinates. The vector $\Delta\hat{X}$ and the parameter λ^2 can assist us in defining which datum is to be preferred among various possible choices.

The external reliability of GPS networks is not affected by the geometrical distribution of the network points since the design matrix A and the normal matrix N are independent of the geometrical shape of the network (Even-Tzur and Papo 1996).

4 Experiments with datum definition

To further explore the relationship between datum definition and reliability a small two-dimensional schematic GPS network, composed of 25 points, was designed (see Fig. 1). Three cases of vector configuration between network points were tested. In case A, 48 vectors that were evenly spread between the network's points were simulated (depicted as solid lines in Fig. 1). In case B, 12 vectors were added in the center of the network (depicted as horizontal broken lines in Fig. 1), and in total 60 vectors were simulated. In case C another 12 vectors were added to the perimeter of the network, totaling 72 vectors (depicted as broken lines in the perimeter of Fig. 1).

The variance (in meters) of a vector of length ℓ_{ij} meters between points i and j was given by $\sigma_{\ell_i} = 0.003 + \ell_{ij} \times 0.5\,ppm$, and a correlation of 10 % was assumed between the two vector components. Zero correlation was assumed between any two different vectors. The weight matrix was produced using a variance of a unit weight equaling one, $\sigma_0^2 = 1$.

The minimum and maximum values of the diagonal element of $\mathcal{H}$ for all cases are presented in Tab. 1. The results
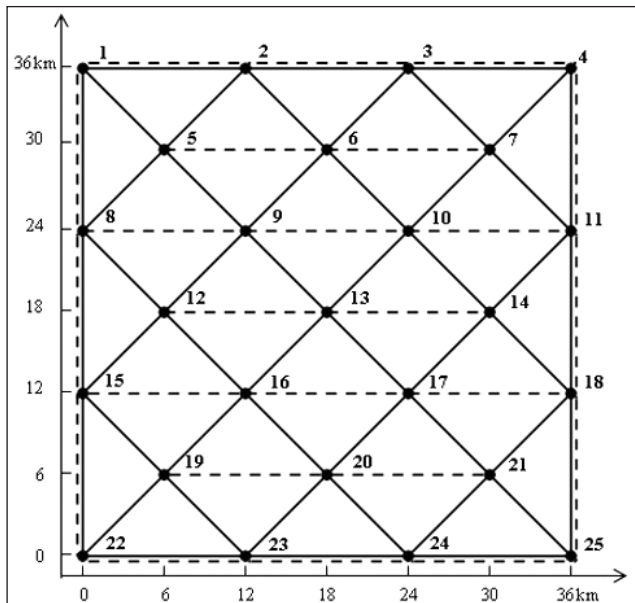
Fig. 1: A two dimensional schematic GPS network composed of 25 points. The simulated vectors are divided into three cases: case A: 48 vectors (solid lines), case B: 60 vectors (solid lines and horizontal broken lines in the center), case C: 72 vectors (all lines).

Tab. 1: The minimum and maximum values of the diagonal element of $\mathcal{H}$

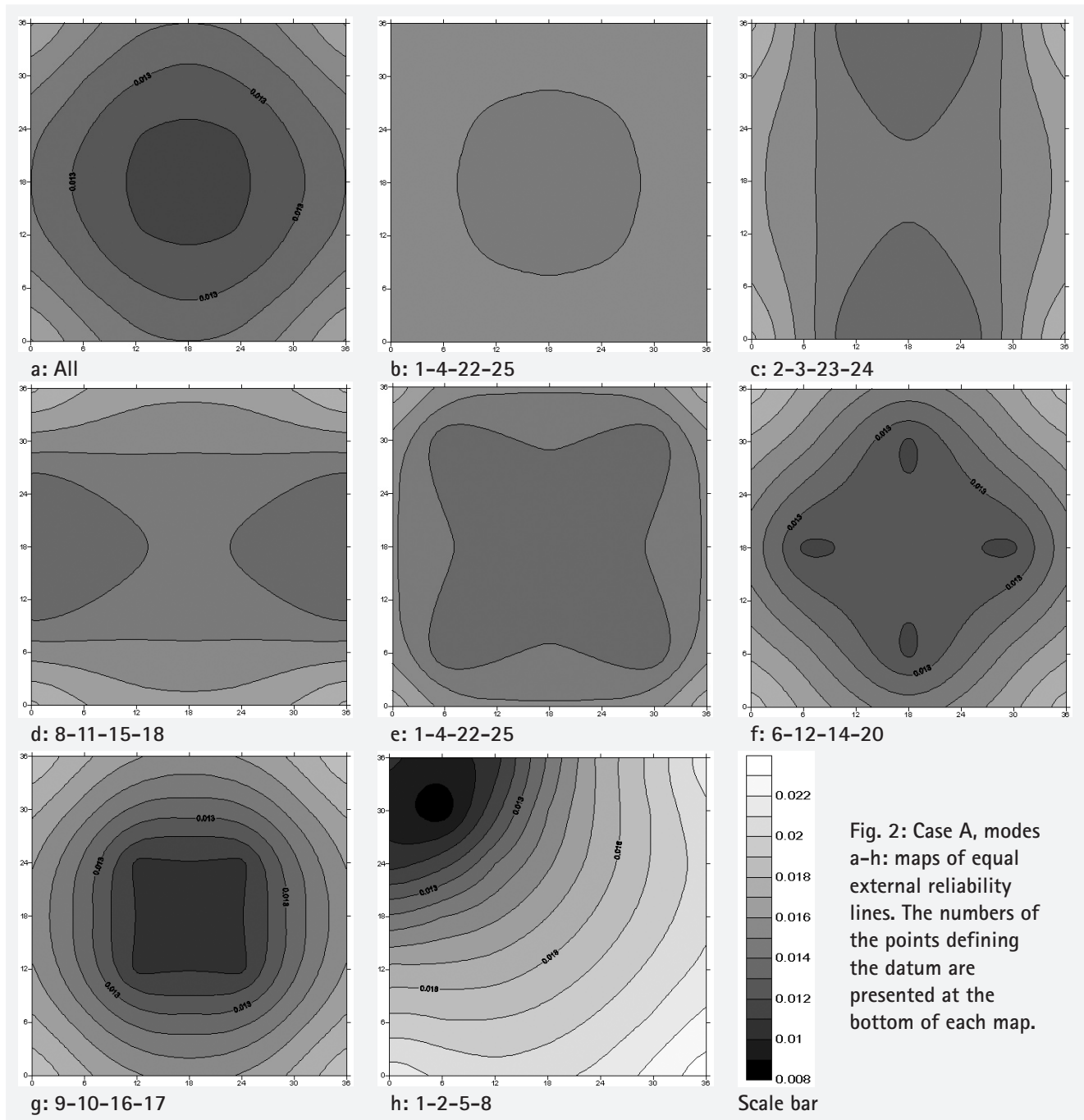| Case	diagonal element of $\mathcal{H}$	
	min	max
A	0.42	0.55
B	0.26	0.54
C	0.25	0.43

Tab. 2: The number of GPS vectors connected to each network point and their degrees of freedom.

Point no.	A (48 vectors) No. of vectors	r_p	B (60 vectors) No. of vectors	r_p	C (72 vectors) No. of vectors	r_p
1	3	1.58	3	1.64	5	3.48
2	4	2.10	4	2.27	6	4.11
3	4	2.10	4	2.27	6	4.11
4	3	1.58	3	1.64	5	3.48
5	4	1.88	5	2.85	5	3.10
6	4	1.92	6	3.73	6	3.90
7	4	1.88	5	2.85	5	3.10
8	4	2.10	5	3.05	7	4.89
9	4	1.92	6	3.85	6	3.92
10	4	1.92	6	3.85	6	3.92
11	4	2.10	5	3.05	7	4.89
12	4	1.92	5	3.02	5	3.14
13	4	1.94	6	3.90	6	3.93
14	4	1.92	5	3.02	5	3.14
15	4	2.10	5	3.05	7	4.89
16	4	1.92	6	3.85	6	3.92
17	4	1.92	6	3.85	6	3.92
18	4	2.10	5	3.05	7	4.89
19	4	1.88	5	2.85	5	3.10
20	4	1.92	6	3.73	6	3.90
21	4	1.88	5	2.85	5	3.10
22	3	1.58	3	1.64	5	3.48
23	4	2.10	4	2.27	6	4.11
24	4	2.10	4	2.27	6	4.11
25	3	1.58	3	1.64	5	3.48

show the following: the internal reliability of the network in case A is not high. The 12 vectors added in case B in the center of the network decreased the minimum value of the internal reliability even more, but there are still some vectors with a low internal reliability. In case C (72 vectors) the achieved internal reliability is reasonable.

Tab. 2 presents the number of GPS vectors connected to each network point and their degrees of freedom for each case.

A total of three cases were simulated for the schematic network. Eight datum configuration modes were defined for each case, one composed of all network points and seven composed of different sets of four points each, where the difference between the four datum points was their geometric distribution. The external reliability for each GPS vector configuration (cases A, B and C) for every datum definition is shown in Tab. 3, 4 and 5. The information about the external reliability is detailed separately for the datum points and the control points by presenting the minimum and maximum values. Additionally, the global external reliability (λ^2) of the datum points, the control points and all other network points is presented. For comparison, the trace of the cofactor matrix is given for each experiment.

Illustrations of the external reliability are presented in Fig. 2, 3 and 4, where maps of equal external reliability lines are presented. Those maps visually demonstrate the propagation of the external reliability per single component within the network. Since the simulated network is symmetric relative to the y and x components, the external reliability values (and the maps) are similar for components y and x.

5 Discussion and Conclusions

Internal reliability can be defined by the diagonal elements of the »hat« matrix $\mathcal{H}$ and does not depend on the datum definition of the network. The internal reliability is a function of the number of observations measured between the network points.

The external reliability, as presented by Baarda, denotes the impact of a single gross error on the adjusted coordinates and depends on the datum definition. However, Baarda's measure for the global external reliability is not suitable for geodetic networks since it does not

a: All

b: 1–4–22–25

c: 2–3–23–24

d: 8–11–15–18

e: 1–4–22–25

f: 6–12–14–20

g: 9–10–16–17

h: 1–2–5–8

Scale bar

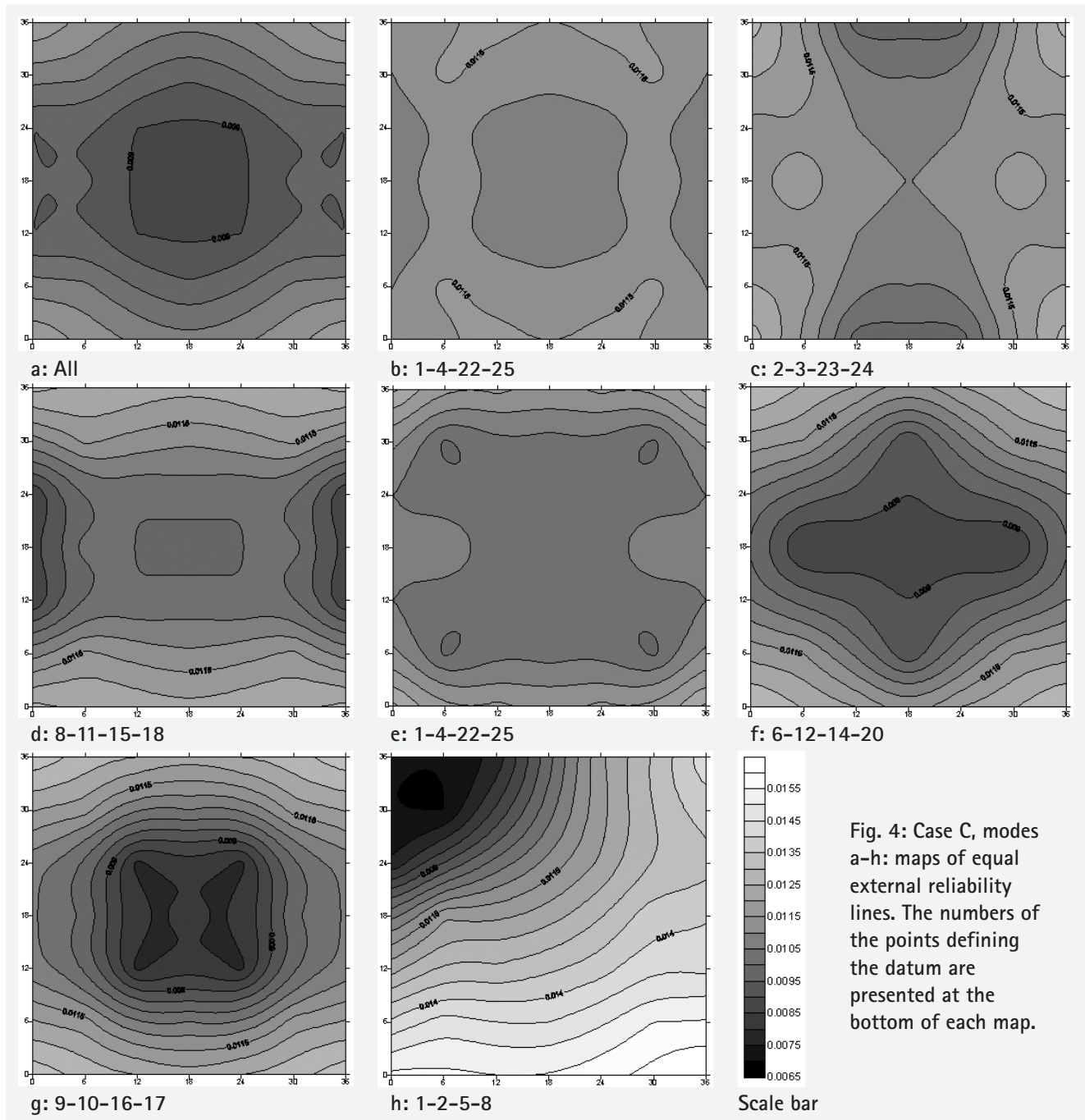Fig. 2: Case A, modes a–h: maps of equal external reliability lines. The numbers of the points defining the datum are presented at the bottom of each map.

Tab. 3: Case A: The external reliability of the network points based on 48 vectors. $\Delta\hat{X}$ is the external reliability vector of the adjusted coordinates, λ is the global external reliability factor and Q is the cofactor matrix.

Mode	Datum defined by points	Datum points			Control points			All points	
		$\Delta\hat{X}$ [cm]		λ [cm]	$\Delta\hat{X}$ [cm]		λ [cm]	λ [cm]	$\sqrt{\mathrm{tr}(Q)}$ [cm]
		min	max		min	max			
a	All	1.15	1.71	9.98	–	–	–	10.0	2.5
b	1–4–22–25	1.59	1.59	4.51	1.46	1.57	9.93	11.9	2.8
c	2–3–23–24	1.30	1.30	3.68	1.35	1.82	10.21	10.9	2.7
d	8–11–15–18	1.30	1.30	3.68	1.35	1.82	10.21	10.9	2.7
e	5–7–19–21	1.33	1.33	3.76	1.32	1.77	9.80	10.5	2.6
f	6–12–14–20	1.18	1.18	3.33	1.24	1.86	10.03	10.6	2.7
g	9–10–16–17	1.08	1.08	3.05	1.05	1.89	10.30	10.7	2.7
h	1–2–5–8	0.84	1.06	2.84	1.40	2.36	12.40	12.7	3.2

Fig. 3: Case B, modes a–h: maps of equal external reliability lines. The numbers of the points defining the datum are presented at the bottom of each map.
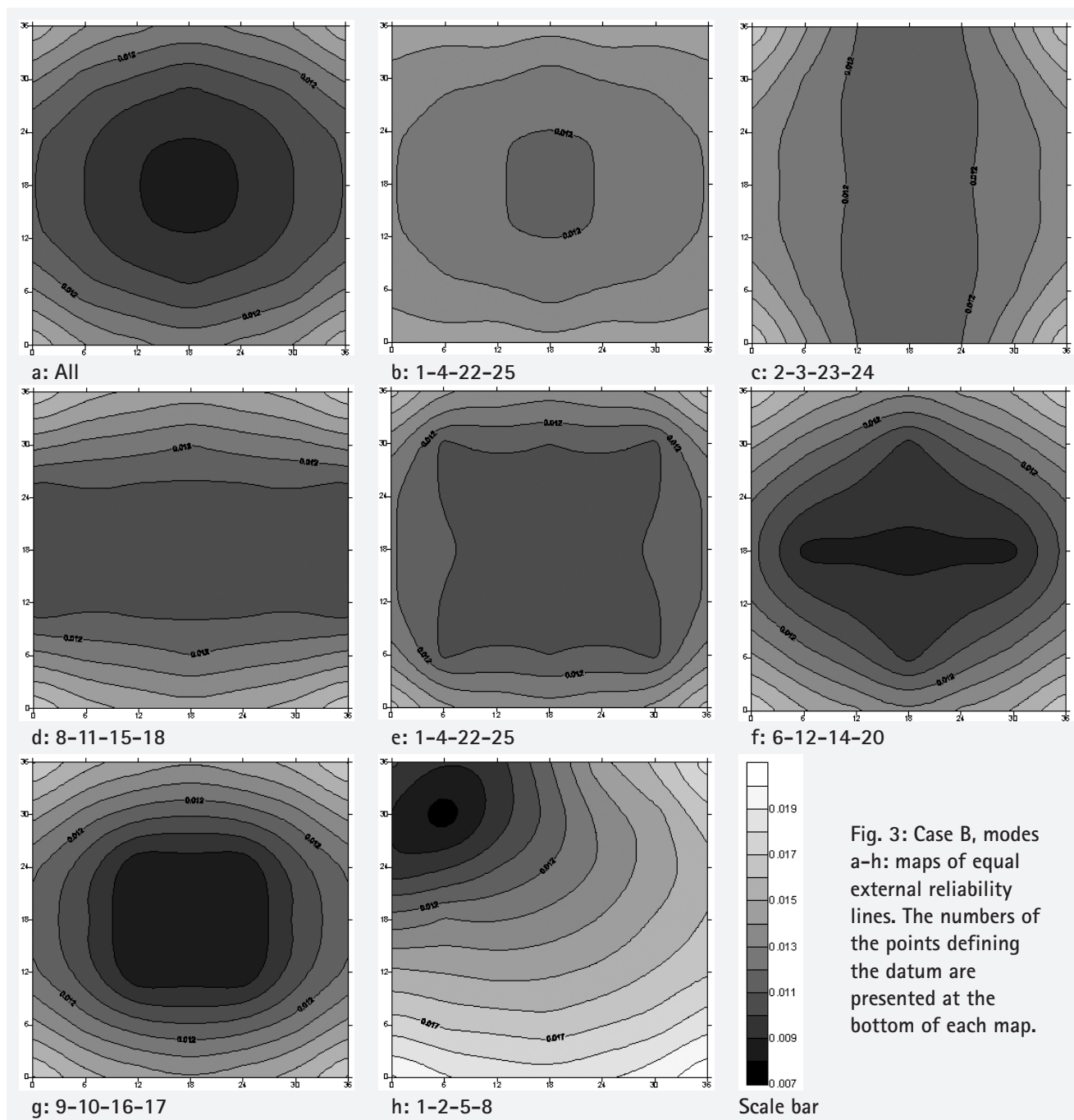
Tab. 4: Case B: The external reliability of the network points based on 60 vectors. $\Delta\hat{X}$ is the external reliability vector of the adjusted coordinates, λ is the global external reliability factor and Q is the cofactor matrix.

Mode	Datum defined by points	Datum points $\Delta\hat{X}$ [cm] min	max	λ [cm]	Control points $\Delta\hat{X}$ [cm] min	max	λ [cm]	All points λ [cm]	$\sqrt{\text{tr}(Q)}$ [cm]
a	All	0.87	1.58	8.45	–	–	–	8.5	2.3
b	1–4–22–25	1.46	1.46	4.13	1.19	1.47	8.49	9.4	2.5
c	2–3–23–24	1.20	1.20	3.41	1.13	1.68	8.71	9.4	2.5
d	8–11–15–18	1.02	1.02	2.88	1.02	1.68	8.64	9.1	2.5
e	5–7–19–21	1.08	1.08	3.06	1.02	1.64	8.35	9.0	2.4
f	6–12–14–20	0.88	0.98	2.63	0.90	1.69	8.50	8.9	2.4
g	9–10–16–17	0.82	0.82	2.31	0.82	1.71	8.71	9.0	2.5
h	1–2–5–8	0.75	1.02	2.61	1.10	2.08	10.38	10.7	2.9

a: All

b: 1–4–22–25

c: 2–3–23–24

d: 8–11–15–18

e: 1–4–22–25

f: 6–12–14–20

g: 9–10–16–17

h: 1–2–5–8

Scale bar

Fig. 4: Case C, modes a–h: maps of equal external reliability lines. The numbers of the points defining the datum are presented at the bottom of each map.

Tab. 5: Case C: The external reliability of the network points based on 72 vectors. $\Delta \hat{X}$ is the external reliability vector of the adjusted coordinates, λ is the global external reliability factor and Q is the cofactor matrix.

Mode	Datum defined by points	Datum points			Control points			All points	
		$\Delta \hat{X}$ [cm] min	max	λ [cm]	$\Delta \hat{X}$ [cm] min	max	λ [cm]	λ [cm]	$\sqrt{tr(Q)}$ [cm]
a	All	0.86	1.19	7.23	–	–	–	7.2	2.1
b	1–4–22–25	1.10	1.10	3.12	1.06	1.16	7.23	7.9	2.3
c	2–3–23–24	0.98	0.98	2.78	1.05	1.27	7.48	8.0	2.3
d	8–11–15–18	0.86	0.86	2.43	0.99	1.26	7.41	7.8	2.2
e	5–7–19–21	0.99	0.99	2.79	1.01	1.26	7.18	7.7	2.2
f	6–12–14–20	0.85	0.91	2.49	0.89	1.30	7.29	7.7	2.2
g	9–10–16–17	0.79	0.79	2.23	0.82	1.33	7.51	7.8	2.3
h	1–2–5–8	0.70	0.77	2.09	1.02	1.65	8.86	9.1	2.6

depend on the datum definition (see equation (9')). The total impact of all marginally detectable gross errors on the adjusted coordinates can be calculated using equation (11) and presented as a vector, which defines the external reliability for every coordinate in the network. An expression for measuring the network's global external reliability is given by equation (12). This expression is based on the network's datum definition and measures the network's global external reliability. The minimal global external reliability for a network is achieved when all the network points are used to define the datum. This corresponds with the approach stating that a datum based evenly on all the points in the network is the optimal datum due to the minimal trace of its cofactor matrix.

Vanícek et al. (2001) take the attitude that reliability analysis should be invariant with respect to datum definition, and could therefore define the robustness of geodetic networks. Their reliability measures are almost independent of the datum definition when the datum of the network is defined with minimum constraints. In practice most geodetic networks rely on a datum specified by an over-constraints solution. This paper claims that reliability measures, which are dependent on datum definitions, are essential to understanding the influence of undetected gross errors on the adjusted coordinates of geodetic networks. Cross (1983) determined the external reliability in a way which saliently depends on datum definition but it is valid for uncorrelated measurements.

Several experiments were conducted on a schematic GPS network in order to investigate the influence of the datum definition on the network reliability, and to investigate the relationship between the geometry of the datum points and the reliability of the network points.

As expected, the best network global external reliability was achieved when all points in the network were used to define the datum, as can be clearly seen from Tab. 3, 4 and 5 (mode a). The improvement in the external reliability, as seen from case A to case B to case C, was the direct outcome of increasing the number of measured vectors, or in other words, when the internal reliability increased the external reliability increased as well.

The external reliability of a point, as shown from the experiments, is a combination of the geometry and the degree of freedom of the datum points. In this study the geometry of the points referred to the size and distribution of the datum points within the network. In the presented schematic GPS network the reliability of the datum points is identical as well, when the degree of freedom of the datum points is identical.

In mode b the datum points used defined the widest datum. In cases A, B, and C those points have a low degree of freedom and the network's global external reliability obtained is low. But in mode b the global external reliability of the control points is high and the maximum value of the external reliability of a single point is the lowest. In that mode the external reliability of the datum points is also relatively high.

In mode g the datum points used defined a narrow datum. In cases A, B and C those points have a high degree of freedom. That combination creates a low external reliability for the control points.

The illustrations in Fig. 2, 3 and 4 clearly demonstrate the relationship between the geometry of the datum points, their degrees of freedom and the network's external reliability when discussing modes b-g.

There are cases, mainly in deformation analysis, where the datum points are defined by a specific subset of points that can not be defined with a wide geometry. In such cases, all the datum points are concentrated in a part of the network. This situation is described in mode h. The size and distribution of the datum points is narrow and the degree of freedom for those points is low. The external reliability distribution is illustrated in Fig. 2, 3 and 4. We can clearly see that the reliability decreases when the points are located farther away from the datum. For example, in the case of point 9 that is located closest to the datum and has a high degree of freedom, and therefore has the minimal external reliability of all the control points. Unlike point 25 that is located farther from the datum and has a low degree of freedom, and therefore has the maximal external reliability of all the control points.

The solutions obtained in all cases clearly display the effect of the datum selection on the reliability of the network. A datum, based on points with a higher reliability, provides a more reliable network.

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