Note on Lars E. Sjöberg:
New Solutions to the Direct and Indirect Geodetic
Problems on the Ellipsoid   (zfv 1/2006, pp. 35 – 39)

Hubert Schmidt

Summary

This note points out some inaccuracies in Sjöberg's solution on the direct and indirect geodetic problems on the ellipsoid. The shortcomings arise in the case of the indirect geodetic problem if the endpoints of the geodesic are located on the equator.

Zusammenfassung

Die von Sjöberg vorgelegte Lösung der ersten und zweiten Hauptaufgabe der Geodätischen Linie auf dem Ellipsoid ist im Falle der Berechnung der zweiten Hauptaufgabe unkorrekt, wenn die Endpunkte der Geodätischen Linie auf dem Äquator liegen.

1   Solutions to the direct and indirect geodetic problems on the ellipsoid

The ›classical‹ solutions to the direct and indirect geodetic problems on the ellipsoid are based on power series. Unfortunately, for different precisions or for different distances, different voluminous power series are required. I have introduced (Schmidt 1999) a solution for the calculation of the direct and indirect geodetic problems that employs a numerical integration (quadrature) with a small set of formulas for all cases. Sjöberg (2006) proposes a strict solution for the sphere with a correction to the ellipsoid determined by numerical integration.

2   Behaviour of the geodesic with endpoints on the equator

2.1   Problems in Sjöberg's solution

In the case of the indirect problem (chapter 3b case iii, if $\beta_1 = \beta_2 = 0°$), Sjöberg's solution leads to incorrect results. This occurs when the endpoints $P_1(\varphi_1, \lambda_1)$ and $P_2(\varphi_2, \lambda_2)$ of the geodesic are located on the equator ($\varphi_1 = \varphi_2 = 0°$).

According to Sjöberg, the Clairaut constant c is given by his equation (14a)

$$c = \frac{\cos(DL)}{\sqrt{t_{11}^2 + \cos^2(DL)}} .$$

In his example 5.2a ($\lambda_1 = \beta_1 = \beta_2 = 0°, \lambda_2 = 50°$), Sjöberg computes the Clairaut constant as $c = 0.999403$. As a consequence, with $\beta_{max} = 1° 58' 48''$ and the azimuth $\alpha_1 = 88° 1' 17''$ the geodesic runs north of the equator.

If this solution was correct, the results would be ambiguous because there would be a second geodesic (with identical but negative β_{max}) running south of the equator. However, geodesics with $\Delta\lambda < 179°$ cannot be ambiguous (see chapter 3). Thus Sjöberg's solution is incorrect in this case.

2.2   Correct solution for the geodesic with endpoints on the equator

On the equator the sphere and the ellipsoid are identical. Thus, the calculation of φ_{max} with the latitudes φ_1 and φ_2 according to Moritz (1959) yield

$$\tan \varphi_{max}$$

$$= \frac{\sqrt{\tan^2 \varphi_1 + \tan^2 \varphi_2 - 2 \tan \varphi_1 \tan \varphi_2 \cos \Delta\lambda}}{\sin \Delta\lambda} \quad (1)$$

$$\tan \beta_{max} = \sqrt{1 - e^2} \tan \varphi_{max} . \quad (2)$$

Alternatively, the direct calculation of β_{max} with the reduced latitudes β_1 and β_2

$$\tan \beta_{max}$$

$$= \frac{\sqrt{\tan^2 \beta_1 + \tan^2 \beta_2 - 2 \tan \beta_1 \tan \beta_2 \cos \Delta\lambda}}{\sin \Delta\lambda} \quad (3)$$

leads – with $\varphi_1 = \varphi_2 = \beta_1 = \beta_2 = 0°$, $\lambda_1 = 0°$, $\lambda_2 = 50°$ – to the same results in which $\varphi_{max} = \beta_{max} = 0°$ and therefore $c = 1$ and azimuth $\alpha_1 = 90°$. That means that the geodesic follows the equator. The distance s between the endpoints of the geodesic can be derived from a simple calculation of the arc of a circle with the radius equal to the semi-major axis a of the ellipsoid:

$$s = a2\pi \frac{(\lambda_2 - \lambda_1) \, rad}{2\pi}$$
$$= a(\lambda_2 - \lambda_1) \, rad \quad (4)$$

with

$$(\lambda_2 - \lambda_1) \, rad = \frac{(\lambda_2 - \lambda_1)°}{180°} \cdot \pi . \quad (5)$$

3 Behaviour of the geodesic in the neighbourhood of diametrically opposed endpoints

Assuming that – for the endpoints $P_1(\varphi_1, \lambda_1)$ and $P_2(\varphi_2, \lambda_2)$ of the geodesic – it holds

$$179° \, 23' \, 36'' < \lambda_2 - \lambda_1 < 180° \tag{6}$$

and simultaneously

$$\varphi_2 + \varphi_1 < 36.4' \, , \tag{7}$$

then the course of the geodesic is ambiguous (Jordan/Eggert/Kneissl 1959, §129). That means that there are two courses of the geodesic with equal distances but different azimuths.

Helmert (1880, p. 329) has defined this ambiguous region exactly. In this region, the calculations of φ_{max} with eq. (1) and β_{max} with eq. (3), respectively, fail, because the denominators tend to zero ($\sin \Delta\lambda \to 0$). The ›classical‹ methods of power series fail as well.

For the first time, I have solved the indirect geodetic problem in this ambiguous region by numerical integration (quadrature) (Schmidt 2000). It has been demonstrated for the ambiguous region that the Clairaut constant c decreases from $c = 1$ to $c = 0$ and β_{max} increases from $\beta_{max} = 0°$ to $\beta_{max} = 90°$. That means that in the case $\beta_1 = \beta_2 = 0°$ and $\lambda_2 - \lambda_1 \leq 179° \, 23' \, 38''$, the geodesic follows the equator. In the ambiguous region (from $\lambda_2 - \lambda_1 \geq 179° \, 23' \, 39''$ to $\lambda_2 - \lambda_1 = 180°$), the geodesic leaves the equator and begins to ›ascend‹ to the north (and south, respectively). Finally, in the case $\lambda_2 - \lambda_1 = 180°$, the geodesic intersects the north pole (and south pole, respectively). The behaviour of the geodesics has been calculated and discussed in Schmidt (2000, Table 3). An extract is given in Tab. 1.

References

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Author's address

Dr.-Ing. Hubert Schmidt
Geodätisches Institut der RWTH Aachen
Templergraben 55, 52062 Aachen
Tel. +49 (0)241 8095281
hubert.schmidt@gia.rwth-aachen.de

Tab. 1: Behaviour of the geodesic with the endpoints $P_1(\varphi_1, \lambda_1)$ and $P_2(\varphi_2, \lambda_2)$ on the equator ($\varphi_1 = \varphi_1 = 0°$) of the International Hayford Ellipsoid

$\lambda_2 - \lambda_1$	β_{max}	distance s [m]	azimuth α_1	azimuth α_2
179° 20'	0° (equator)	19 964 080.968	90°	270°
179° 23' 38''	0° (equator)	19 970 822.247	90°	270°
179° 23' 39''	+ 01° 34' 04''	19 970 853.166	88° 25' 56''	271° 34' 04''
	– 01° 34' 04''	– '' –	91° 34' 04''	268° 25' 56''
179° 30'	+ 34° 23' 20''	19 981 603.278	55° 36' 40''	304° 23' 20''
	– 34° 23' 20''	– '' –	124° 23' 20''	235° 36' 40''
179° 59'	+ 88° 25' 22''	20 004 551.065	01° 34' 38''	358° 25' 22''
	– 88° 25' 22''	– '' –	178° 25' 22''	181° 34' 38''
180°	+ 90° (meridian)	20 004 576.598	0°	360°
	– 90° (meridian)	– '' –	180°	180°

Comment to H. Schmidt's Remarks on Sjöberg, zfv 1/2006, pp. 35 – 39

Lars E. Sjöberg

In general, H. Schmidt is right that Eq. (14a) of Sjöberg (2006) with the two given points located on the equator leads to wrong results (except for the longitude difference close to π). However, this cannot be proved by Eq. (1) [nor by Eq. (2)] of Schmidt's note, because this equation is only valid on the sphere. (For two given points on the equator of the sphere, the equator is the great circle connecting the two points whenever the longitude difference differs from π. In the latter case any great circle is a geodesic for the sphere.)

For the ellipsoid $\Delta\lambda$ of Schmidt's Eq. (1) should be substituted by $D\lambda$, or the equation should be modified to (Sjöberg, in press)

$$\tan\beta_{max} = \frac{\sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos(D\lambda)}}{\sin(D\lambda)}, \tag{1}$$

where $t_i = \tan\beta_i$ and $D\lambda = \Delta\lambda - d\lambda$. Here $d\lambda$ was given by Eq. (6b) of Sjöberg (2006). For two given points located on the same latitude Eq. (1) can be written

$$\tan\beta_{max} = t_1 / \cos(D\lambda/2), \tag{2}$$

and

$$d\lambda = -2e^2 cI$$
$$= -2e^2 c \int_{\sin\beta_1}^{\sqrt{1-c^2}} \frac{dx}{\sqrt{1-c^2-x^2}\left[1+\sqrt{1-e^2(1-x^2)}\right]}. \tag{3}$$

If $D\lambda \neq \pi$, Eq. (2) yields, in the limiting case with the two given points located on the equator, $t_{max} = t_1 = 0$, i. e. the Clairaut constant c equals 1. However, this result does not hold for $D\lambda = \pi$, i. e. for the longitude difference with $|\pi - \Delta\lambda| \leq e^2\pi/2$ (approximately), in which case the denominator of Eq. (2) becomes 0. In this case the Clairaut constant follows from Eq. (3) and the equation $\pi = \Delta\lambda - d\lambda$. The result is

$$c = \frac{\pi - \Delta\lambda}{2e^2 I} \approx \frac{2}{e^2}\left(1 - \frac{\Delta\lambda}{\pi}\right), \tag{4}$$

where $I \approx \pi/4$ is the integral of Eq. (3), which shows that c decreases from 1 to 0 as $\Delta\lambda$ goes from (approximately) $\pi(1 \pm e^2/2) = 180° \pm 36'09''$ (for e^2 set to 6.694×10^{-3}) and approaches π.

»Ne nouveau sous le soleil«; the old great masters like F. R. Helmert already knew the correct solutions from a theoretical point of view. By taking advantage of today's technology we may only facilitate some computations. Of course, this must be done properly with care.

References

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Author's address
Lars E Sjöberg
Royal Institute of Technology, Division of Geodesy
SE-100 44 Stockholm, Sweden
Phone: +46 8 790-7330, Fax: + 46 8 790-7343
sjoberg@geomatics.kth.se